

# Boundary $S$ matrices with $N = 2$ supersymmetry

Rafael I. Nepomechie

Physics Department, P.O. Box 248046, University of Miami

Coral Gables, FL 33124 USA

## Abstract

We propose the exact boundary  $S$  matrix for breathers of the  $N = 2$  supersymmetric sine-Gordon model. We argue that this  $S$  matrix has three independent parameters, in agreement with a recently-proposed action. We also show, contrary to a previous claim, that the “universal” supersymmetric boundary  $S$  matrix commutes with two supersymmetry charges. General  $N = 2$  supersymmetric boundary integrable models are expected to have boundary  $S$  matrices with a similar structure.

# 1 Introduction

Much is known about bulk [1] and boundary [2]  $S$  matrices of two-dimensional integrable models. Bulk  $S$  matrices  $S(\theta)$  for integrable models with  $N = 1$  or  $N = 2$  supersymmetry have the product structure

$$S(\theta) = S_{Bose}(\theta) S_{SUSY}(\theta), \quad (1)$$

where  $S_{Bose}(\theta)$  is the  $N = 0$  bulk  $S$  matrix, which is purely Bosonic; and the “universal”  $S$  matrix  $S_{SUSY}(\theta)$  is an 8-vertex (6-vertex)  $R$  matrix satisfying the so-called free-Fermion condition [3], for the case  $N = 1$  ( $N = 2$ ). See, e.g., [4]-[9] for  $N = 1$ , and [7],[10]-[15] for  $N = 2$ , respectively. One expects that the corresponding boundary  $S$  matrices  $\mathbf{S}(\theta)$  should have a similar structure,<sup>1</sup>

$$\mathbf{S}(\theta) = \mathbf{S}_{Bose}(\theta) \mathbf{S}_{SUSY}(\theta), \quad (2)$$

where  $\mathbf{S}_{Bose}(\theta)$  and  $\mathbf{S}_{SUSY}(\theta)$  satisfy boundary Yang-Baxter equations with the corresponding bulk  $S$  matrices  $S_{Bose}(\theta)$  and  $S_{SUSY}(\theta)$ , respectively. Indeed, for  $N = 1$ , this is precisely what has been found [16]-[21]. However, for  $N = 2$ , the situation has been less clear: the “universal”  $\mathbf{S}_{SUSY}(\theta)$  has been claimed [22] to commute with only one supersymmetry charge, and has therefore been rejected in favor of more complicated  $S$  matrices with nontrivial boundary structure.

We argue here that this  $\mathbf{S}_{SUSY}(\theta)$  *does* in fact commute with two supersymmetry charges; and hence, the structure (2) holds also for the  $N = 2$  case. Although this result is rather general, for definiteness and simplicity we focus here on the particular case of the first breathers of the  $N = 2$  supersymmetric sine-Gordon model.

The outline of this Letter is as follows. In Section 2 we propose the bulk and boundary  $S$  matrices for the breathers of the  $N = 2$  sine-Gordon model. Although the boundary  $S$  matrix appears to depend on four boundary parameters, we show that there is one relation among them. Hence, there are only three independent boundary parameters, in agreement with the recently-proposed action [23]. In Section 3, we verify that the boundary  $S$  matrix has  $N = 2$  supersymmetry. We conclude with a brief discussion in Section 4.

## 2 The $N = 2$ sine-Gordon model

Actions for the bulk and boundary  $N = 2$  sine-Gordon (SG) model have been constructed in [13] and [23], respectively. Substantial evidence has been given that both the bulk and boundary versions of the model are integrable, and we now assume that this is indeed the case. The work [14, 15] suggests that the breathers form two-dimensional irreducible representations of the  $N = 2$  supersymmetry algebra. Following [15], we denote these one-particle states by  $u(\theta)$  and  $d(\theta)$ , where  $\theta$  is the rapidity.<sup>2</sup> These states have Fermion

---

<sup>1</sup>We make an effort to distinguish boundary quantities from the corresponding bulk quantities by using sans serif letters to denote the former, and Roman letters to denote the latter.

<sup>2</sup>As usual, we set  $E = m \cosh \theta$ ,  $P = m \sinh \theta$ , where  $m$  is the particle mass.

number  $\frac{1}{2}$  and  $-\frac{1}{2}$ , respectively. For simplicity, we restrict our attention to the first (lowest-mass) breathers. In view of (1), a natural conjecture for the bulk (two-particle)  $S$  matrix  $S_{SG\ b}^{N=2}(\theta, \beta^{N=2})$  is (see also [14, 15])

$$S_{SG\ b}^{N=2}(\theta, \beta^{N=2}) = S_{SG\ b}^{N=0}(\theta, \beta^{N=0}) S_{SUSY}(\theta), \quad (3)$$

where  $\theta$  is the difference in rapidity of the two particles, and the scalar factor  $S_{SG\ b}^{N=0}(\theta, \beta^{N=0})$  is the  $N = 0$  sine-Gordon breather  $S$  matrix [24, 1]

$$S_{SG\ b}^{N=0}(\theta, \beta^{N=0}) = \frac{\sinh \theta + i \sin \frac{\gamma}{8}}{\sinh \theta - i \sin \frac{\gamma}{8}}, \quad (4)$$

where  $\gamma = \beta_{N=0}^2 / (1 - (\beta_{N=0}^2 / 8\pi))$ . The universal  $N = 2$  bulk  $S$  matrix  $S_{SUSY}(\theta)$  is the sine-Gordon soliton  $S$  matrix [1] with  $\beta^2 = \frac{16\pi}{3}$ ,

$$S_{SUSY}(\theta) = S_{SG\ s}^{N=0}(\theta, \beta^2 = \frac{16\pi}{3}) = Y(\theta)R(\theta), \quad (5)$$

where  $R(\theta)$  is the  $4 \times 4$  matrix

$$R(\theta) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad (6)$$

with matrix elements

$$a = i \cosh \frac{\theta}{2}, \quad b = \sinh \frac{\theta}{2}, \quad c = i. \quad (7)$$

The scalar factor  $Y(\theta)$  is given by

$$Y(\theta) = \frac{1}{i \cosh \frac{\theta}{2}} \exp \left( \frac{i}{2} \int_0^\infty \frac{dt}{t} \frac{\sin(t\theta)}{\cosh^2 \frac{\pi t}{2}} \right). \quad (8)$$

Finally,  $\beta^{N=2}$  is the dimensionless bulk coupling constant appearing in the action, which is related to  $\beta^{N=0}$  by [14, 15]

$$\beta_{N=2}^2 = \gamma = \frac{\beta_{N=0}^2}{1 - \frac{\beta_{N=0}^2}{8\pi}}. \quad (9)$$

For the corresponding breather boundary  $S$  matrix  $S_{SG\ b}^{N=2}(\theta, \beta^{N=2})$ , we propose

$$S_{SG\ b}^{N=2}(\theta, \beta^{N=2}) = S_{SG\ b}^{N=0}(\theta, \beta^{N=0}; \tilde{\eta}, \tilde{\vartheta}) S_{SUSY}(\theta; \eta, \vartheta), \quad (10)$$

where the scalar factor  $S_{SG\ b}^{N=0}(\theta, \beta^{N=0}; \tilde{\eta}, \tilde{\vartheta})$  is the  $N = 0$  sine-Gordon breather boundary  $S$  matrix [25]

$$\begin{aligned} S_{SG\ b}^{N=0}(\theta, \beta^{N=0}; \tilde{\eta}, \tilde{\vartheta}) &= \frac{\cosh(\frac{\theta}{2} + \frac{i\gamma}{32}) \cosh(\frac{\theta}{2} - \frac{i\pi}{4} - \frac{i\gamma}{32}) \sinh(\frac{\theta}{2} + \frac{i\pi}{4})}{\cosh(\frac{\theta}{2} - \frac{i\gamma}{32}) \cosh(\frac{\theta}{2} + \frac{i\pi}{4} + \frac{i\gamma}{32}) \sinh(\frac{\theta}{2} - \frac{i\pi}{4})} \\ &\times \frac{\left( \cos(\frac{\gamma\tilde{\eta}}{8\pi}) + i \sinh \theta \right) \left( \cosh(\frac{\gamma\tilde{\vartheta}}{8\pi}) + i \sinh \theta \right)}{\left( \cos(\frac{\gamma\tilde{\eta}}{8\pi}) - i \sinh \theta \right) \left( \cosh(\frac{\gamma\tilde{\vartheta}}{8\pi}) - i \sinh \theta \right)}. \end{aligned} \quad (11)$$

Moreover,  $S_{SUSY}(\theta; \eta, \vartheta)$  is the sine-Gordon soliton boundary  $S$  matrix [2] with  $\beta^2 = \frac{16\pi}{3}$ ,

$$S_{SUSY}(\theta; \eta, \vartheta) = S_{SG}^{N=0}(\theta, \beta^2 = \frac{16\pi}{3}; \eta, \vartheta) = Y(\theta; \eta, \vartheta) R(\theta; \eta, \vartheta), \quad (12)$$

where  $R(\theta; \eta, \vartheta)$  is the  $2 \times 2$  matrix

$$R(\theta; \eta, \vartheta) = \begin{pmatrix} \mathcal{A}_+ & \mathcal{B} \\ \mathcal{B} & \mathcal{A}_- \end{pmatrix}, \quad (13)$$

with matrix elements

$$\mathcal{A}_\pm = \cos\left(\xi \mp \frac{i\theta}{2}\right), \quad \mathcal{B} = -\frac{ik}{2} \sinh \theta. \quad (14)$$

The pair of boundary parameters  $(\xi, k)$  is related to  $(\eta, \vartheta)$  by

$$\cos \eta \cosh \vartheta = -\frac{1}{k} \cos \xi, \quad \cos^2 \eta + \cosh^2 \vartheta = 1 + \frac{1}{k^2}. \quad (15)$$

An explicit expression for the scalar factor  $Y(\theta; \eta, \vartheta)$ , which we shall not need here, can be obtained from [2]. As usual [1, 2], the  $S$  matrices (3), (10) may contain additional CDD-like factors.

The boundary  $S$  matrix (10) apparently depends on four boundary parameters:  $\tilde{\eta}, \tilde{\vartheta}, \eta, \vartheta$ . However, they are not all independent. For example, let us suppose that  $\tilde{\eta}$  lies in the range

$$\frac{4\pi^2}{\gamma} < \tilde{\eta} < \frac{8\pi^2}{\gamma}. \quad (16)$$

The scalar factor (11) has a pole at  $\theta = iv$  with  $v = \frac{\gamma\tilde{\eta}}{8\pi} - \frac{\pi}{2}$ , which then lies in the physical strip and corresponds to a boundary bound state. We recall [2] the following general constraint: near a pole  $iv_{0a}^\alpha$  of the boundary  $S$  matrix associated with the excited boundary state  $|\alpha\rangle_B$  (which can be interpreted as a boundary bound state of particle  $A_a$  with the boundary ground state  $|0\rangle_B$ ), the boundary  $S$  matrix must have the form

$$S_a^b(\theta) \simeq \frac{i}{2} \frac{g_{a0}^\alpha g_\alpha^{b0}}{\theta - iv_{0a}^\alpha}, \quad (17)$$

where  $g_{a0}^\alpha$  are boundary-particle couplings. For the boundary  $S$  matrix (10), this implies <sup>3</sup>

$$(\mathcal{A}_+ \mathcal{A}_- - \mathcal{B}^2) \Big|_{\theta=iv} = 0. \quad (18)$$

Remarkably, we find (after some computation) that this constraint reduces to simply  $v = 2\eta \pm \pi$ . <sup>4</sup> It follows that

$$\eta = \frac{\gamma\tilde{\eta}}{16\pi} + \frac{\pi}{4}, \quad (19)$$

and so there are indeed only three independent boundary parameters. (For values of  $\tilde{\eta}$  outside the range (16), one must presumably consider  $S$  matrices for the higher breathers and/or solitons.) The fact that the boundary action [23] has the same number of parameters lends support to the proposed expression (10) for the boundary  $S$  matrix. In the next section, we provide further support by demonstrating that this  $S$  matrix has  $N = 2$  supersymmetry.

<sup>3</sup>There is a similar relation for the  $N = 1$  case [20, 21].

<sup>4</sup>There is another solution  $v = 2i\vartheta \pm \pi$ , which we discard for  $\vartheta$  real.

### 3 $N = 2$ supersymmetry of the $S$ matrix

Following [15], we assume that the supersymmetry charges  $Q^\pm, \bar{Q}^\pm$  act on the one-particle states as follows:

$$\begin{aligned} Q^-|u(\theta)\rangle &= \sqrt{2m}e^{\frac{\theta}{2}}|d(\theta)\rangle, & \bar{Q}^+|u(\theta)\rangle &= \sqrt{2m}e^{-\frac{\theta}{2}}|d(\theta)\rangle, \\ Q^+|d(\theta)\rangle &= \sqrt{2m}e^{\frac{\theta}{2}}|u(\theta)\rangle, & \bar{Q}^-|d(\theta)\rangle &= \sqrt{2m}e^{-\frac{\theta}{2}}|u(\theta)\rangle, \end{aligned} \quad (20)$$

and otherwise annihilate the states. The action on multi-particle states is specified further by the coproduct

$$\begin{aligned} \Delta(Q^\pm) &= Q^\pm \otimes 1 + e^{\pm i\pi F} \otimes Q^\pm, \\ \Delta(\bar{Q}^\pm) &= \bar{Q}^\pm \otimes 1 + e^{\mp i\pi F} \otimes \bar{Q}^\pm, \end{aligned} \quad (21)$$

where  $F$  is the Fermion number operator,

$$F|u(\theta)\rangle = \frac{1}{2}|u(\theta)\rangle, \quad F|d(\theta)\rangle = -\frac{1}{2}|d(\theta)\rangle. \quad (22)$$

One can verify that (20)-(22) indeed provide a representation of the  $N = 2$  supersymmetry algebra [26]

$$\begin{aligned} \{Q^+, Q^-\} &= 2(H + P), & \{\bar{Q}^+, \bar{Q}^-\} &= 2(H - P), \\ \{Q^+, \bar{Q}^+\} &= 2m\mathcal{N}, & \{Q^-, \bar{Q}^-\} &= 2m\mathcal{N}, \\ Q^{\pm 2} = \bar{Q}^{\pm 2} &= \{Q^\pm, \bar{Q}^\mp\} = 0, & \{Q^\pm, F\} = \{\bar{Q}^\mp, F\} &= 0, \end{aligned} \quad (23)$$

where  $H, P$  and  $\mathcal{N}$  are the Hamiltonian, momentum and number operators, respectively. One can also verify that the bulk  $S$  matrix (3) commutes with the supersymmetry charges  $Q^\pm, \bar{Q}^\pm$ , as well as with  $F$ .

We can now address the important question: how much supersymmetry does the boundary  $S$  matrix (12) have? Evidently, this  $S$  matrix does not commute with any of the supersymmetry charges or with the Fermion number operator. But it is not difficult to prove that there are two linear combinations of these operators with which the boundary  $S$  matrix *does* commute:

$$\begin{aligned} \hat{Q}^+ &= Q^+ + \bar{Q}^+ + \kappa^+ F, \\ \hat{Q}^- &= Q^- + \bar{Q}^- + \kappa^- F, \end{aligned} \quad (24)$$

where  $\kappa^\pm = \pm i2\sqrt{2m}e^{\pm i\xi}/k$ . This means that the boundary  $S$  matrix has  $N = 2$  supersymmetry. The terms in (24) proportional to  $F$  correspond to local Fermionic boundary terms [23]. Similar Fermionic boundary terms also appear in the  $N = 1$  case [20, 21]. It was the

failure to consider such terms in [22] that led to the erroneous conclusion that the  $S$  matrix has only  $N = 1$  supersymmetry.

We remark that  $\hat{Q}^\pm$  generate the subalgebra

$$\hat{Q}^{\pm 2} = 2m\mathcal{N} + \kappa^{\pm 2}F^2, \quad \left\{ \hat{Q}^+, \hat{Q}^- \right\} = 4H + 2\kappa^+ \kappa^- F^2. \quad (25)$$

We also note that combinations of the form  $Q^\pm + a_\pm \overline{Q}^\mp + b_\pm F$  generally do not commute with the boundary  $S$  matrix.

## 4 Discussion

We have argued that the boundary  $S$  matrix of the  $N = 2$  sine-Gordon model has the structure (2), with the universal  $S$  matrix  $\mathbf{S}_{SUSY}(\theta)$  given by (12). On the basis of established corresponding bulk results (1), we expect that this structure is characteristic of all boundary integrable models with  $N = 2$  supersymmetry.

An important outstanding problem is to find the precise relation between the parameters of the boundary  $S$  matrix and those of the boundary action. This problem has already been addressed for the  $N = 0$  case [27, 28].

Whereas for the  $N = 1$  case there is only one boundary parameter in the universal  $S$  matrix  $\mathbf{S}_{SUSY}$ , for the  $N = 2$  case there are two. Hence,  $N = 2$  models generally should manifest much richer boundary phenomena. It should be particularly interesting to explore the consequences of this for open string theory.

## Acknowledgments

Discussions with L. Mezincescu and A.B. Zamolodchikov in 1995 on related problems are gratefully acknowledged. This work was supported in part by the National Science Foundation under Grant PHY-9870101.

## References

- [1] A.B. Zamolodchikov and A.I. Zamolodchikov, *Ann. Phys.* *120* (1979) 253; A.B. Zamolodchikov, *Sov. Sci. Rev. A2* (1980) 1.
- [2] S. Ghoshal and A.B. Zamolodchikov, *Int. J. Mod. Phys. A9* (1994) 3841.
- [3] C. Fan and F.Y. Wu, *Phys. Rev. B2* (1970) 723.
- [4] R. Shankar and E. Witten, *Phys. Rev. D17* (1978) 2134.

- [5] A.B. Zamolodchikov, “Fractional-spin integrals of motion in perturbed conformal field theory,” in *Fields, Strings and Quantum Gravity*, eds. H. Guo, Z. Qiu and H. Tye, (Gordon and Breach, 1989).
- [6] K. Schoutens, Nucl.Phys. *B344* (1990) 665.
- [7] D. Bernard and A. LeClair, Phys. Lett. *247B* (1990) 309.
- [8] C. Ahn, D. Bernard and A. LeClair, Nucl. Phys. *B346* (1990) 409.
- [9] C. Ahn, Nucl. Phys. *B354* (1991) 57; *B422* (1994) 449.
- [10] P. Fendley, S.D. Mathur, C. Vafa and N.P. Warner, Phys. Lett. *243B* (1990) 257.
- [11] P. Fendley, W. Lerche, S.D. Mathur and N.P. Warner, Nucl. Phys. *B348* (1991) 66.
- [12] P. Mathieu and M.A. Walton, Phys. Lett. *254B* (1991) 106.
- [13] K. Kobayashi and T. Uematsu, Phys. Lett. *B264* (1991) 107.
- [14] K. Kobayashi and T. Uematsu, Phys. Lett. *B275* (1992) 361.
- [15] P. Fendley and K. Intriligator, Nucl. Phys. *B372* (1992) 533; *B380* (1992) 265.
- [16] L. Chim, Int. J. Mod. Phys. *A11* (1996) 4491.
- [17] M. Moriconi and K. Schoutens, Nucl. Phys. *B487* (1997) 756.
- [18] C. Ahn and C. Rim, J. Phys. *A32* (1999) 2509.
- [19] C. Ahn and R.I. Nepomechie, Nucl. Phys. *B586* (2000) 611.
- [20] C. Ahn and R.I. Nepomechie, Nucl. Phys. *B594* (2001) 660.
- [21] R.I. Nepomechie, Phys. Lett. *B509* (2001) 183.
- [22] N.P. Warner, Nucl. Phys. *B450* (1995) 663.
- [23] R.I. Nepomechie, “The boundary  $N = 2$  supersymmetric sine-Gordon model,” hep-th/0106207.
- [24] I.Ya. Aref’eva and V.E. Korepin, JETP Lett. *20* (1974) 312.
- [25] S. Ghoshal, Int. J. Mod. Phys. *A9* (1994) 4801.
- [26] E. Witten and D. Olive, Phys. Lett. *78B* (1978) 97.
- [27] A.B. Zamolodchikov, unpublished work reported at the 1999 Bologna CFT workshop.
- [28] A. Chenaghlou and E. Corrigan, Int. J. Mod. Phys. *A15* (2000) 4417; E. Corrigan and A. Taormina, J. Phys. *A33* (2000) 8739.